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Lagrangian studies of plasma wave interactions II

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Abstract. The lagrangian approach described in part I is used to examine the interaction of positive and negative energy waves. The explosive interaction between two positive ion acoustic waves and a negative energy Bernstein mode is discussed and a growth rate for the instability calculated using the lagrangian method.

1. Introduction

Low (1958) first drew attention to the possible merits of a lagrangian formulation of the Vlasov equation in tackling problems in plasma physics but observed that this formulation involved as much algebraic complication in producing dispersion relations for plasma oscillations and hydromagnetic waves as the conventional approach. It was left to Suramlishvili (1964, 1965, 1967) and Vedenov (1967) to develop this approach to study three- and four-plasmon interaction processes and in these nonlinear developments the lagrangian formulation offers very real computational advantages over the alternate method starting from the Maxwell–Vlasov equations. These studies of wave–wave interactions are discussed in a quantum mechanical formulation which has only become popular outside the Soviet Union in recent years with the translation of the work by Tsytovich (1970) and the excellent review paper by Harris (1969). The lagrangian approach to problems involving nonlinear dispersive waves was developed independently by Whitham in a fluid mechanics context and from this starting point several authors (Dougherty 1970, Dewar 1970) have extended Whitham's ideas to hydromagnetics and plasma dynamics. Others (Galloway and Kim 1971, Boyd and Turner 1972, to be referred to as I) have used Low's Lagrangian for a warm plasma to discuss various wave–wave interactions. Galloway and Kim considered the nonlinear coupling between three colinearly-propagating electrostatic waves in a warm plasma while in I the nonlinear interaction of transverse waves to produce plasma oscillations and of three electromagnetic waves were used as illustrations of the method. Further examples are given by Turner (1972).

In this paper we examine the interaction between positive and negative energy waves (§ 2) and in § 3 use the lagrangian approach to calculate the growth rate for an explosive instability in a warm magnetized plasma. In § 4 a general discussion of the lagrangian approach to studying wave interaction phenomena in warm plasmas is given.

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2. Wave interactions involving negative energy waves

All waves involved in the interactions discussed in I were positive energy waves. Since the total energy of the waves is conserved in each interaction, bounded solutions of the coupled mode equations result. However, if the interaction of positive and negative energy waves is considered, an instability may result provided certain criteria are met. By negative energy waves we mean the following. The energy per unit volume of an electrostatic wave travelling in a plasma with dielectric function $\epsilon^{L}(\boldsymbol{k}, \omega)$ is

$$(|\boldsymbol{E}|^2/8\pi)\{\partial/\partial\omega(\omega\epsilon^{L}(\boldsymbol{k}, \omega))\}_{\omega=\omega_n}$$

where $\boldsymbol{E}$ is the electric field amplitude and $\omega_n > 0$ is a solution of the dispersion relation $\epsilon^{L}(\boldsymbol{k}, \omega) = 0$. Hence if there exist waves in the plasma such that

$$\frac{\partial}{\partial\omega}(\omega\epsilon^{L}(\boldsymbol{k}, \omega))_{\omega=\omega_n} < 0$$

these may be described as having *negative energy* and clearly may only propagate in a dielectric medium.

In interactions between positive and negative energy waves it may happen that the negative energy modes lose energy to the positive energy modes and this transfer of energy results in an increase in the absolute value of the negative energy and a corresponding gain in energy for the positive energy modes. Thus the amplitudes of the interacting waves all grow with time, and the system of waves is unstable. The amplitudes of the interacting waves become infinite after a finite time and such an instability is known as an *explosive instability* (Sagdeev and Galeev 1969).

The *total* energy of the interacting waves is still constant so that the system though unstable in the nonlinear regime is linearly stable.

Consider an interacting triad of waves governed by synchronism conditions

$$\omega_1 + \omega_2 = \omega_3, \qquad \boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 \tag{1}$$

with $\omega_j > 0$ ($j = 1, 2, 3$). The energy density associated with wave n is given by

$$\epsilon_n = \Gamma_n \hat{E}_n \hat{E}_n^* \tag{2}$$

where $\hat{E}_n$ is the electric field amplitude of wave n, so that the sign of the energy will be determined by the sign of Γ_n. From I equation (1) together with the definitions of χ and $\hat{A}_n$, for time variation only, the coupled mode equations are

$$\begin{aligned} \frac{\mathrm{i}}{\omega_{1,2}} \frac{\partial \hat{E}_{1,2}}{\partial t} &= -\frac{\Gamma_{\mathrm{wc}}^*}{\Gamma_{1,2}} \hat{E}_{2,1}^* \hat{E}_3, \\ \frac{\mathrm{i}}{\omega_3} \frac{\partial \hat{E}_3}{\partial t} &= -\frac{\Gamma_{\mathrm{wc}}}{\Gamma_3} \hat{E}_1 \hat{E}_2. \end{aligned} \tag{3}$$

Defining

$$A_j = \left(\frac{|\Gamma_j|}{\omega_j}\right)^{1/2} \hat{E}_j$$

$$V = \left(\frac{\Gamma_{\mathrm{wc}}}{\mathrm{i}}\right)\left(\frac{\omega_1\omega_2\omega_3}{|\Gamma_1\Gamma_2\Gamma_3|}\right)^{1/2}$$

together with the signature $S_n = \pm 1$ where $S_n = \Gamma_n/|\Gamma_n|$, the coupled-mode equations become

$$\frac{\partial A_{1,2}}{\partial t} = S_{1,2} V A^*_{2,1} A_3,$$
$$\frac{\partial A_3}{\partial t} = -S_3 V A_1 A_2. \tag{4}$$

Thus negative energy waves will be characterized by negative S_n. To examine stability criteria differentiate (4) once with respect to time to obtain

$$\frac{\partial^2 A_1}{\partial t^2} = V^2(S_1 S_2 |A_3|^2 - S_1 S_3 |A_2|^2) A_1$$
$$\frac{\partial^2 A_2}{\partial t^2} = V^2(S_1 S_2 |A_3|^2 - S_2 S_3 |A_1|^2) A_2 \tag{5}$$
$$\frac{\partial^2 A_3}{\partial t^2} = -V^2(S_2 S_3 |A_1|^2 + S_1 S_3 |A_2|^2) A_3.$$

Since $V^2 \geqslant 0$ and $|A_j|^2 \geqslant 0$, we observe that an instability will occur if

$$S_1 = S_2 = -S_3 \tag{6}$$

in which case the amplitudes of the three waves will all increase with time. It is clear from (6) that an instability occurs if the wave with highest frequency has energy differing in sign from the two lower frequency waves. The amplitudes of all three waves can grow monotonically with time, that is, the modes become explosively unstable.

Solutions of the coupled-mode equations follow using the standard approach (cf Sagdeev and Galeev 1969). Writing

$$A_j(t) = a_j(t) \exp\{-\mathrm{i}(\omega_j t + \psi_j(t))\}$$

where the amplitudes and phases are $a_j(t)$, $\psi_j(t)$ respectively and are both real, the solution of the coupled-mode equations leads to two Manley–Rowe relations

$$a_3^2 + S_2 S_3 a_2^2 = \mu_1,$$
$$a_3^2 + S_1 S_3 a_1^2 = \mu_2 \tag{7}$$

together with

$$a_1 a_2 a_3 \sin\theta = \sigma \tag{8}$$

where μ_1, μ_2, σ are constants of integration, and $\theta = (\psi_3 - \psi_2 - \psi_1)$. The solution for a_3^2 may then be shown to be

$$2Vt = \int_{a_3^2(0)}^{a_3^2(t)} \frac{\mathrm{d}(a_3^2)}{\{(a_3^2 - \alpha_1)(a_3^2 - \alpha_2)(a_3^2 - \alpha_3)\}^{1/2}} \tag{9}$$

where $\alpha_1, \alpha_2, \alpha_3$, are the roots of the bicubic

$$Q \equiv a_3^2(a_3^2 \oplus \mu_1)(a_3^2 \oplus \mu_2) - \sigma^2 = 0. \tag{10}$$

The sign $\oplus$ indicates that for the explosive regime ($S_1 = S_2 = -S_3$) the plus sign is taken, whilst for stable solutions ($S_1 = S_2 = S_3$) the minus sign is understood. In the case where Q possesses three distinct roots, ($0 \leqslant \alpha_1 < \alpha_2 < \alpha_3$) stable solutions for the three wave amplitudes are

$$\begin{aligned} a_3^2(t) &= \alpha_1+(\alpha_2-\alpha_1)\,\mathrm{sn}^2\{V(\alpha_3-\alpha_1)^{1/2}(t-t_0),\gamma\} \\ a_2^2(t) &= a_3^2(0)+a_2^2(0)-a_3^2(t) \\ a_1^2(t) &= a_3^2(0)+a_1^2(0)-a_3^2(t) \end{aligned} \tag{11}$$

where

$$\gamma = \left(\frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1}\right)^{1/2}$$

is the modulus of the jacobian elliptic function sn and t_0 is a constant.

For an explosive interaction solutions of the coupled-mode equations are

$$\begin{aligned} a_3^2(t) &= \alpha_1+\frac{\alpha_3-\alpha_1}{\mathrm{sn}^2\{V(\alpha_3-\alpha_1)^{1/2}(t_0-t),\gamma\}} \\ a_2^2(t) &= a_3^2(t)-a_3^2(0)+a_2^2(0) \\ a_1^2(t) &= a_3^2(t)-a_3^2(0)+a_1^2(0) \end{aligned} \tag{12}$$

from which it can be seen that $a_3^2(t) \to \infty$ as $t \to t_0$, so that the amplitudes of the interacting waves become infinite after a time t_0, the explosion time, given by

$$t_0 = \{V(\alpha_3-\alpha_1)^{1/2}\}^{-1}\,\mathrm{sn}^{-1}\left\{\left(\frac{\alpha_3-\alpha_1}{a_3^2(0)-\alpha_1}\right)^{1/2},\gamma\right\}. \tag{13}$$

A nonlinear growth rate $\gamma_{\mathrm{explosive}}$ for the instability may be defined by the reciprocal of t_0,

$$\gamma_{\mathrm{explosive}} \simeq V(\alpha_3-\alpha_1)^{1/2}.$$

Taking $\sigma = 0$ it follows from (10) that $\alpha_1 = 0$ and $\alpha_3 = \mu_1 = a_3^2$. The nonlinear growth rate is then

$$\gamma_{\mathrm{explosive}} = -\mathrm{i}\Gamma_{\mathrm{wc}}\left(\frac{\omega_1\omega_2\omega_3}{|\Gamma_1\Gamma_2\Gamma_3|}\right)^{1/2}(\omega_3^{-1}\mathscr{E}_3)^{1/2} \tag{14}$$

where $\mathscr{E}_3$ denotes the energy density per unit volume in wave 3.

3. Interaction of positive and negative energy waves in a magnetized plasma

Equation (14) determines the nonlinear growth rate in terms of the coupling coefficient Γ_{wc}. In this section we apply the lagrangian method to compute $\gamma_{\mathrm{explosive}}$ for a particular instability involving the interaction of ion acoustic waves (1, 2) with a negative energy Bernstein mode (3) so that, from (6), we satisfy the criterion for an explosive instability. Electron Bernstein modes only possess negative energy for frequencies in a range $0 < \omega < kv_0$ where v_0 is a drift velocity of electrons relative to the ions. We consider such waves and apply the lagrangian method to obtain first the dispersion relation (and hence the Γ_j) and then compute Γ_{wc}.

The variation of $\mathscr{L}_2$ with respect to $\boldsymbol{r}$ yields (cf I)

$$m_\alpha \mathscr{D}_\alpha^2 \boldsymbol{r}_\alpha = q_\alpha\left(\boldsymbol{E}^{(1)}+\frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}^{(1)}\right)+\frac{q_\alpha}{c}\mathscr{D}_\alpha \boldsymbol{r}_\alpha\times\boldsymbol{B}_0 \tag{15}$$

where $\boldsymbol{E}^{(1)}$, $\boldsymbol{B}^{(1)}$ represent the first order electric and magnetic fields respectively, $\boldsymbol{B}_0$ is the zero order constant magnetic field and α is the species label. The equations of motion for the electrons and ions become

$$m_e \mathscr{D}^2 \boldsymbol{r}_e = -e\boldsymbol{E} - \frac{e}{c}\mathscr{D}\boldsymbol{r}_e \times \boldsymbol{B}_0 \tag{16}$$

$$m_i \mathrm{D}^2 \boldsymbol{r}_i = e\boldsymbol{E} \tag{17}$$

respectively where

$$\mathscr{D} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla - \frac{e}{mc}\boldsymbol{v} \times \boldsymbol{B}_0 \cdot \frac{\partial}{\partial \boldsymbol{v}}$$

$$\equiv \mathrm{D} - \frac{e}{mc}\boldsymbol{v} \times \boldsymbol{B}_0 \cdot \frac{\partial}{\partial \boldsymbol{v}}$$

and we have dropped the superscript from the first order electric field. The solutions of (16) and (17) follow immediately from a standard analysis giving

$$\boldsymbol{r}_e = \frac{eE}{2m_e} \sum_{n,m=-\infty}^{\infty}\sum (\omega - n\Omega)^{-1} \mathrm{J}_m\left(\frac{kv_\perp}{\Omega}\right) \mathrm{J}_n\left(\frac{kv_\perp}{\Omega}\right) \exp\{\mathrm{i}(n-m)\psi\} (\hat{\boldsymbol{x}} V_n + \hat{\boldsymbol{y}} \mathrm{i} W_n) \tag{18}$$

$$\boldsymbol{r}_i = \frac{-e\boldsymbol{E}}{m_i(\omega - \boldsymbol{k} \cdot \boldsymbol{v})^2} \tag{19}$$

where

$$V_n, W_n = (\omega - \overline{n+1}\Omega)^{-1} \pm (\omega - \overline{n-1}\Omega)^{-1}$$

$$\boldsymbol{v} = (v_\perp \cos\psi, v_\perp \sin\psi, v_z)$$

$$\Omega = \frac{eB_0}{mc}$$

is the electron cyclotron frequency. We have taken $\boldsymbol{k} = (k, 0, 0)$, $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$. The dispersion relation is found by taking the variation of $\mathscr{L}_2$ with respect to ϕ, that is

$$\nabla \cdot \boldsymbol{E} = -4\pi \sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v} f_{0\alpha}(\nabla \cdot \boldsymbol{r}_\alpha). \tag{20}$$

The zero order distribution functions are

$$f_{0i} = n_0 \left(\frac{m_i}{2\pi\kappa T_i}\right)^{3/2} \exp\left\{-m_i\left(\frac{v^2}{2\kappa T_i}\right)\right\}$$

$$f_{0e} = n_0 \left(\frac{m_e}{2\pi\kappa T_e}\right)^{3/2} \exp\left(\frac{-m_e\{(v_x - v_0)^2 + v_y^2 + v_z^2\}}{2\kappa T_e}\right)$$

and the dispersion relation is

$$1 - \frac{\omega_{pi}^2}{\omega^2} - \lambda^{-1}\left(\frac{\omega_{pe}}{\Omega}\right)^2 \sum_{n=-\infty}^{\infty} \frac{n\Omega\, \mathrm{e}^{-\lambda} \mathrm{I}_n(\lambda)}{(\omega - kv_0) - n\Omega} = 0 \tag{21}$$

where $\omega_{p\alpha}^2 = 4\pi n_0 q_\alpha^2/m_\alpha$, $\lambda = k^2\kappa T_e/m_e\Omega^2$ and I_n is a modified Bessel function of the first kind. Since $\Sigma_{n=-\infty}^{\infty} \mathrm{e}^{-\lambda}\mathrm{I}_n(\lambda) = 1$, the infinite sum in (21) can be transformed and the dispersion relation becomes

$$\epsilon^{L}(\boldsymbol{k},\omega) = 1-\frac{\omega_{pi}^2}{\omega^2}-\frac{\omega_{pe}^2}{k^2V_e^2}\left(-1+e^{-\lambda}I_0(\lambda)+(\omega-kv_0)^2\sum_{n=1}^{\infty}\frac{2e^{-\lambda}I_n(\lambda)}{(\omega-kv_0)^2-n^2\Omega^2}\right) = 0 \qquad (22)$$

where $V_e^2 = \kappa T_e/m_e$. For plasma parameters such that $\lambda \gg 1$, if $\omega - kv_0 \simeq -n\Omega$, we can approximate the infinite sum by one term so that (22) becomes

$$\left(\omega^2-\frac{k^2c_s^2}{1+k^2\lambda_D^2}\right)\{(\omega-kv_0)^2-n^2\Omega^2\} = 2\omega^2(\omega-kv_0)^2e^{-\lambda}I_n(\lambda) \qquad (23)$$

where $c_s^2 = \kappa T_e/m_i$ is the ion acoustic speed and λ_D is the Debye length. Since $\lambda \gg 1$, $e^{-\lambda}I_n(\lambda) \sim (2\pi\lambda)^{-1/2}$ so that the right hand side of (23) is much less than one. Hence, for large λ, the motions of the ion acoustic waves and the Bernstein modes are effectively decoupled. It is clear from the definition in § 2 using (22) that Bernstein modes with frequencies in the range $0 < \omega < kv_0$ are waves of negative energy. We now consider the explosive interaction of two positive energy ion acoustic waves and a negative energy Bernstein mode.

3.1. *Calculation of $\gamma_{\text{explosive}}$ for the Bernstein mode–ion acoustic waves triad*

The synchronism conditions for the interaction are

$$\begin{aligned}\omega_1^{IA}+\omega_2^{IA} &= \omega_3^{B},\\ \boldsymbol{k}_1^{IA}+\boldsymbol{k}_2^{IA} &= \boldsymbol{k}_3^{B}\end{aligned} \qquad (24)$$

with $0 < \omega_3^B < k_3^B v_0$ and $\omega_3 - k_3v_0 \simeq -n\Omega$. Since

$$\Gamma_n = (8\pi)^{-1}\left(\frac{\partial}{\partial\omega}(\omega\epsilon^L(\boldsymbol{k},\omega))\right)_{\omega=\omega_n}$$

use of the dispersion relation (22) enables the various Γ_n to be written immediately

$$8\pi\Gamma_{IA} = \frac{2\omega_{pi}^2}{\omega_{IA}^2} \qquad (25)$$

$$8\pi\Gamma_B = -\left(\frac{\omega_B}{n\Omega\beta_n}\right)\{1+(k_B\lambda_D)^2\}^2 \qquad (26)$$

where $\beta_n = \exp(-\lambda)I_n(\lambda)$.

For three electrostatic modes, the third order Lagrangian $\mathscr{L}_3$ is given by

$$\mathscr{L}_{3\alpha} = -q_\alpha f_{0\alpha}\{\tfrac{1}{2}(\boldsymbol{r}_\alpha\cdot\nabla)^2\phi\}$$

so that on separating $\boldsymbol{r}$ and ϕ into their wave components

$$\boldsymbol{r} = \boldsymbol{r}_1+\boldsymbol{r}_2+\boldsymbol{r}_3, \qquad \phi = \phi_1+\phi_2+\phi_3$$

and space–time averaging (cf I) we find

$$\mathscr{L}_3 = \tfrac{1}{8}\sum_\alpha q_\alpha f_{0\alpha}(\hat{\boldsymbol{r}}_{1\alpha}\cdot\hat{\boldsymbol{r}}_{2\alpha}k_3^2\hat{\phi}_3^*+\hat{\boldsymbol{r}}_{3\alpha}^*\cdot\hat{\boldsymbol{r}}_{1\alpha}k_2^2\hat{\phi}_2+\hat{\boldsymbol{r}}_{3\alpha}^*\cdot\hat{\boldsymbol{r}}_{2\alpha}k_1^2\hat{\phi}_1)+\text{cc}. \qquad (27)$$

The coupling coefficient Γ_{wc} may then be determined directly from (27) and I (21, 27*c*) giving

$$\Gamma_{wc} = \frac{n_0}{8}\left(\frac{ie^3k_3}{m_e^2k_1^2k_2^2v_e^4} - \frac{ie^3k_3}{m_i^2\omega_1^2\omega_2^2}\right) - \frac{n_0}{16}\left(\frac{ie^3(k_1^3+k_2^3)}{m_e^2k_1^2k_2^2v_e^2}\sum_{n=-\infty}^{\infty}\frac{e^{-\lambda}I_n(\lambda)V_{n3}}{\omega_3 - k_3v_0 - n\Omega}\right) \tag{28}$$

where V_{n3} is given by V_n where $\omega \to \omega_3$ etc. Retaining the dominant third term of (28) and using the dispersion relation to replace the infinite sum in (28) by $2/\omega_{pe}^2$ and considering parameters such that $k_j\lambda_D \sim 1$ $(j = 1, 2)$ reduces (28) to the simple form

$$\Gamma_{wc} = i/32\pi^{3/2}(n_0\kappa T_e)^{-1/2}. \tag{29}$$

Substituting (25), (26) and (29) into (14) gives

$$\gamma_{explosive} \sim \frac{2^{1/2}}{4}(n\Omega\omega_3\beta_n)^{1/2}\left(\frac{\mathscr{E}_3}{n_0\kappa T_e}\right)^{1/2}. \tag{30}$$

A similar growth rate has been obtained for the interaction between two negative energy Bernstein modes (1, 2) and an ion acoustic wave (3) where $\omega_1 + \omega_2 = \omega_3$, $\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3$ (Boyd and Turner 1971). It has been suggested that these processes may be important in certain situations of practical interest such as collisionless shocks. However, a calculation of nonlinear wave interactions based on the well defined phase approach is not likely to provide a valid description of phenomena in the turbulent plasma found in a collisionless shock. For such plasmas a random phase approach is preferable and, as one would expect, this gives a smaller growth rate, proportional to $(\mathscr{E}_3/n_0\kappa T_e)$ rather than $(\mathscr{E}_3/n_0\kappa T_e)^{1/2}$.

Growth rates such as (30) may, however, be of interest in themselves if experiments on these interactions were possible. At present there appears to be little, if any, laboratory evidence for explosive instabilities. There is of course a growing body of experimental data on wave–wave interactions. For example the decay of a large amplitude Bernstein mode into an ion acoustic wave and a second Bernstein mode† has been observed by Keen and Fletcher (1971).

4. Discussion

In this paper and in I we have had the restricted aim of examining the value of a lagrangian approach in wave–wave interactions in plasmas. The lagrangian density $\mathscr{L}$ was developed in a perturbation series describing the linear wave spectrum ($\mathscr{L}_2$), three-wave interactions ($\mathscr{L}_3$), four-wave interactions ($\mathscr{L}_4$) etc. At the linear stage $\int \bar{\mathscr{L}}_2\, d\boldsymbol{v} = 0$ (the bar denoting the space–time averaging described in I) gives the dispersion relation for the waves and, as Low (1958) pointed out, no reduction in algebraic effort is achieved by the lagrangian formalism. In the case of nonlinear interactions on the other hand there is real gain and the lagrangian method is to be preferred over the perturbation theory approach starting from the equations of motion, which has been standard in the literature for some time. Until recently this does not appear to have been appreciated in plasma physics except for some work by Suramlishvili (1964, 1967) and Vedenov (1967) on three- and four-plasmon processes in a quantum mechanical formulation.

As an example, consider three-plasmon processes in an isotropic plasma in which one ion acoustic (s) plasmon and two electron plasma (l) plasmons take part. Starting

† Γ_{wc} for this decay is simply obtained by the lagrangian formula (Turner 1972).

from the classical Lagrangian $\mathscr{L}_3$ and introducing second quantization leads to a set of kinetic equations for the distribution functions of the l plasmons n_k and the s plasmons N_k:

$$\dot{n}_1 = \sum W_{12}\{-n_1(n_3+1)(N_2+1)+(n_1+1)n_3N_2\}$$
$$+\sum W_{21}\{-n_1(n_3+1)N_2+(n_1+1)n_3(N_2+1)\} \tag{31}$$

where W_{12} is the probability for the decay $\mathrm{l}(\boldsymbol{k}_1) \to \mathrm{s}(\boldsymbol{k}_2)+\mathrm{l}(\boldsymbol{k}_3)$ and W_{21} the probability for the combination $\mathrm{l}(\boldsymbol{k}_1)+\mathrm{s}(\boldsymbol{k}_2) \to \mathrm{l}(\boldsymbol{k}_3)$. The summation in (31) is taken over wave numbers k_2, k_3 satisfying $\boldsymbol{k}_1 = \boldsymbol{k}_2+\boldsymbol{k}_3$ in the first term and $\boldsymbol{k}_1+\boldsymbol{k}_2 = \boldsymbol{k}_3$ in the second. For the s plasmons

$$\dot{N}_2 = \sum W_{21}\{-N_2n_1(n_3+1)+(N_2+1)(n_1+1)n_3\} \tag{32}$$

with W_{21} the probability for the combination $\mathrm{s}(\boldsymbol{k}_2)+\mathrm{l}(\boldsymbol{k}_1) \to \mathrm{l}(\boldsymbol{k}_3)$. There is no decay contribution in (32) since the frequency conservation relation cannot be satisfied. The probabilities in (31) and (32) are expressed in terms of the matrix elements of the Lagrangian $\int \bar{\mathscr{L}}_3 \, \mathrm{d}\boldsymbol{v}$

$$W_{12} = \frac{2\pi}{\hbar}\left|\left\langle n_1, N_2, n_3 \left| \int \bar{\mathscr{L}}_3 \, \mathrm{d}\boldsymbol{v} \right| n_1-1, N_2+1, n_3+1 \right\rangle\right|^2$$
$$W_{21} = \frac{2\pi}{\hbar}\left|\left\langle n_1, N_2, n_3 \left| \int \bar{\mathscr{L}}_3 \, \mathrm{d}\boldsymbol{v} \right| n_1-1, N_2-1, n_3+1 \right\rangle\right|^2.$$

Suramlishvili (1967) has also obtained kinetic equations for three- and four-plasmon processes in an anisotropic plasma.

Thus whether we consider nonlinear wave interactions in the well defined phase approximation (as in § 2) or in the random phase approximation used in deriving the kinetic equations (31), (32) the coupling constants or matrix elements for the interactions are most simply computed in a lagrangian formulation. It is hardly surprising that a method which leads directly to conservation laws provides these coefficients more efficiently than via the Vlasov–Maxwell equations (cf Rohrlich 1965).

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